

ON THE UNRAMIFIED PRINCIPAL SERIES OF $\mathrm{GL}(3)$ OVER NON-ARCHIMEDEAN LOCAL FIELDS

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ABSTRACT. Let F be a non-archimedean local field and let \mathcal{O} be its ring of integers. We give a complete description of the irreducible constituents of the restriction of the unramified principal series representations of $\mathrm{GL}_3(F)$ to $\mathrm{GL}_3(\mathcal{O})$.

1. INTRODUCTION

1.1. Overview. Let F be a non-archimedean local field with ring of integers \mathcal{O} , maximal ideal \mathfrak{P} and let π be a fixed uniformizer. Our interest in this paper is in the restriction of unramified principal series representations of $\mathrm{GL}_n(F)$ to its maximal compact subgroup $\mathrm{GL}_n(\mathcal{O})$. More precisely, let B be a Borel subgroup of GL_n and let T be a maximal torus contained in B . Concretely, we may take the group of upper triangular matrices and the group of diagonal matrices, respectively. Any linear character $\chi : T(F) \rightarrow \mathbb{C}^\times$ can be inflated to a linear character of $B(F)$, still denoted χ . The principal series representation corresponding to χ is the induced representation $\mathrm{Ind}_{B(F)}^{\mathrm{GL}_n(F)}(\chi)$ of $\mathrm{GL}_n(F)$ on the space of continuous functions

$$V_\chi = \{f \in C(\mathrm{GL}_n(F)) \mid f(bg) = \chi(b)\|b\|^{1/2}f(g), \text{ for all } g \in \mathrm{GL}_n(F), b \in B(F)\},$$

where $b \mapsto \|b\|$ is the modular character of $B(F)$. We assume that the representation is unramified, namely, that the restriction of χ to $T(\mathcal{O})$ is trivial, and focus on the decomposition to irreducible constituents of its restriction to $\mathrm{GL}_n(\mathcal{O})$. By Frobenius reciprocity this restriction is isomorphic to $\mathrm{Ind}_{B(\mathcal{O})}^{\mathrm{GL}_n(\mathcal{O})}(\mathbf{1})$. The case $n = 2$ was fully treated in [2]. Partial results for the case $n = 3$ were obtained in [1] and it is the goal of this paper to give a complete description in this case. Further results on the restriction of principal series representations of GL_n to the maximal compact subgroup can be found in [3].

1.2. The unramified principal series of $\mathrm{GL}(3)$. We first describe and reformulate several results concerning GL_3 which were obtained by Campbell and Nevins in [1]. Let $G = \mathrm{GL}_3(\mathcal{O})$, let B be its subgroup of upper triangular matrices and let $V = \mathrm{Ind}_B^G(\mathbf{1})$. For $\ell \in \mathbb{N}$, let K^ℓ denote the ℓ^{th} principal congruence subgroup of G , namely, the kernel of the canonical map from G to $\mathrm{GL}_3(\mathcal{O}/\mathfrak{P}^\ell)$. Being normal subgroups of G , the groups K^ℓ give rise to a coarse decomposition of V into G -invariant spaces $V \cong \bigoplus_{\ell=1}^{\infty} V^{K^\ell}/V^{K^{\ell-1}}$. A finer decomposition of these spaces, yet not to irreducible ones, is obtained by considering certain ‘parabolic’ subgroups of G which are defined as follows. Let

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\mathbb{N}_0 stand for $\mathbb{N} \cup \{0\}$ endowed with the natural ordering and consider \mathbb{N}_0^3 with the product ordering, i.e. $c \preceq d$ if and only if $c_i \leq d_i$ for $i = 1, 2, 3$. Let C be the cone

$$C = \{(c_1, c_2, c_3) \in \mathbb{N}_0^3 \mid c_1, c_2 \leq c_3 \leq c_1 + c_2\}.$$

To an element $c = (c_1, c_2, c_3) \in C$ we associate the compact open subgroup of G

$$P_c = \begin{bmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{P}^{c_1} & \mathcal{O} & \mathcal{O} \\ \mathcal{P}^{c_3} & \mathcal{P}^{c_2} & \mathcal{O} \end{bmatrix} \cap G,$$

which contains B . The defining inequalities of C ensure that P_c is indeed a group. Let $U_c = \text{Ind}_{P_c}^G(\mathbf{1})$ be the permutation representation of G arising from its action on the coset space G/P_c . It follows that $P_c \subseteq P_d$ if and only if $c \succcurlyeq d$, hence U_d is a sub-representation of U_c if and only if $d \preceq c$. In the center of our discussion lie the representations

$$V_c = U_c / \sum_{d \prec c} U_d, \quad (c \in C).$$

We have

$$(1.1) \quad V = \bigoplus_{c \in C} V_c.$$

The mutual relations between the representations V_c are completely determined in [1]. To describe them we introduce the auxiliary functions $\lambda, \kappa, \mu : C \rightarrow \mathbb{N}_0$. For $c \in C$ let

$$\begin{aligned} \lambda(c) &= c_3, \\ \kappa(c) &= c_1 + c_2 - c_3, \\ \mu(c) &= \min\{c_1 + c_2 - c_3, c_3 - c_1, c_3 - c_2\}. \end{aligned}$$

With these we have

(CN1) Let $c, d \in C$. Then

$$V_c \simeq V_d \text{ if and only if } \lambda(c) = \lambda(d), \kappa(c) = \kappa(d), \mu(c) = \mu(d) \text{ and } c = d \text{ if } \kappa(c) > \mu(c).$$

Statement (CN1) is a reformulation of several results from [1] which requires a proof (see Proposition 6.2 below). Let C° and ∂C denote the interior and boundary of C , respectively. That is

$$\begin{aligned} C^\circ &= \{c = (c_1, c_2, c_3) \mid c_1, c_2 < c_3 < c_1 + c_2\}, \\ \partial C &= C \setminus C^\circ. \end{aligned}$$

Concerning the irreducibility of the representations V_c one has

(CN2) The representation V_c is irreducible if and only if $c \in \partial C$.

Statement (CN2) is proved in [1, Theorems 6.1, 7.1 and 8.1]. The complete description of the double coset spaces $P_c \backslash G / P_d$ ($c, d \in C$), obtained in [1], does not lend itself to decompose the representations V_c ($c \in C^\circ$), which comprise most of the constituents of V . However, crucial for us

is the following observation made in [1, §8]. Let $\rho = (1, 1, 1)$. Let $m \in \mathbb{N}$ and $c \in C^\circ \cap C + m\rho$. For all $d \in [c - m\rho, c]$ let

$$(1.2) \quad \begin{aligned} U_d^m &= \text{Ind}_{P_d}^{P_{c-m\rho}}(\mathbf{1}), \\ V_c^m &= U_c^m / \sum_{c-m\rho \preccurlyeq d \prec c} U_d^m. \end{aligned}$$

It follows that $V_c = \text{Ind}_{P_{c-m\rho}}^G(V_c^m)$. For the special case $m = \mu(c)$ the relations between V_c and V_c^m become very tight.

(CN3) Let $c \in C^\circ$. Then $\dim \text{End}_G(V_c) = \dim \text{End}_{P_{c-\mu(c)\rho}}(V_c^{\mu(c)})$. In particular, the irreducible constituents of V_c are induced from the irreducible constituents of $V_c^{\mu(c)}$.

We remark that any linear combination of the invariants λ , κ and μ above could be used throughout. We made this particular choices as they have natural interpretations: $\lambda(c)$ is the level of the representation V_c , and both $\kappa(c)$, $\mu(c)$ turn out to be the parameters which control the decomposition of V_c ($c \in C^\circ$).

1.3. Description of results and organization of the paper. The main result of this paper is a complete description of the irreducible constituents of the representations V_c ($c \in C^\circ$). These are described in terms of induced linear characters of certain subquotients of the groups P_c . Consequently we obtain a complete description of the irreducibles in V . In Section 2 we define twisted Heisenberg groups and their toral extensions. As it turns out, these are quotients of P_c ($c \in C^\circ$) which carry the complete information on the representations V_c but are much more accessible. In Section 3 we define and analyze in detail specific multiplicity free representations of toral extensions of twisted Heisenberg groups and in Section 4 we show that the V_c^m 's are subrepresentations of these multiplicity free representations. It remains to identify what are the components of V_c^m and this is achieved in Section 5. We end the paper by computing the multiplicities and dimensions of the irreducible constituents in V . Notably, these depend only on the residue field \mathcal{O}/\mathcal{P} , and the dependence is through substitution in universal polynomials defined over \mathbb{Z} .

1.4. Notation, conventions and tools. For $m \in \mathbb{N}$ we write \mathcal{O}_m for the finite quotient $\mathcal{O}/\mathcal{P}^m$. We use $\text{val}(\cdot)$ to denote the valuation on \mathcal{O} . It is convenient to use the same symbol for the finite quotients with the convention that $\text{val}(0) = m$ for $0 \in \mathcal{O}_m$. When chances for confusion are slim we use the same notation for elements or subsets of \mathcal{O} and their respective images in \mathcal{O}_m . For a group G and elements $g, x \in G$ we write ${}^g x = gxg^{-1}$ for the left conjugation action and $x^g = g^{-1}xg$ for right conjugation. If $H < G$ we write ${}^g H = gHg^{-1}$. If χ is a character of H we write ${}^g \chi$ for the character of ${}^g H$ defined by ${}^g \chi(x) = \chi(g^{-1}xg)$ for all $x \in {}^g H$. Throughout we use fairly standard tools from representation theory such as Mackey and Clifford theories and the following criterion for the existence of non-trivial intertwining operators. If χ_i are linear characters of subgroups H_i of G we have the following realization of the intertwining operators

$$\text{Hom}_G(\text{Ind}_{H_1}^G(\chi_1), \text{Ind}_{H_2}^G(\chi_2)) = \{f : G \rightarrow \mathbb{C} \mid f(h_2gh_1) = \chi_2(h_2)f(g)\chi_1(h_1), \forall h_i \in H_i, \forall g \in G\},$$

and an element $g \in G$ supports a non-zero intertwining function if and only if $\chi_1 = {}^g \chi_2$ on $H_1 \cap {}^g H_2$.

2. TWISTED HEISENBERG GROUPS AND THEIR TORAL EXTENSIONS

To study the representation V_c^m we first need to analyze U_c^m , which is the permutation representation of $P_{c-m\rho}$ arising from its action on the right cosets $P_{c-m\rho}/P_c$ for $m \in \mathbb{N}$, $c \in C^\circ \cap C + m\rho$.

Definition 2.1. Let R be a commutative ring with identity and $\delta \in R$. The δ -twisted Heisenberg groups over R , denoted H_R^δ , is the set of triples in R^3 endowed with the multiplication

$$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + \delta yx').$$

A toral extension of a δ -twisted Heisenberg group is the semidirect product $H_R^\delta \rtimes (R^\times)^3$ of $(R^\times)^3$ and H_R^δ , with the action

$$(2.1) \quad {}^t(x, y, z) = (t_1^{-1}t_2x, t_2^{-1}t_3y, t_1^{-1}t_3z),$$

for $t = (t_1, t_2, t_3) \in (R^\times)^3$ and $(x, y, z) \in H_R^\delta$. We denote this semidirect product by B_R^δ .

The family $\{H_R^\delta \mid \delta \in R\}$ interpolate between the standard Heisenberg group ($\delta = 1$) and the abelian group R^3 with addition ($\delta = 0$). The group $B_R^{\delta=1}$ is isomorphic to the group of invertible triangular matrices over R . It is convenient to keep the analogy with this special case and use a matrix presentation for B_R^δ :

$$(2.2) \quad [(x, y, z), (t_1, t_2, t_3)] \mapsto \begin{bmatrix} 1 & & \\ x & 1 & \\ z & y & 1 \end{bmatrix} \circ_\delta \begin{bmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{bmatrix},$$

with multiplication defined by

$$(2.3) \quad \begin{bmatrix} t_1 & & \\ x & t_2 & \\ z & y & t_3 \end{bmatrix} \circ_\delta \begin{bmatrix} t'_1 & & \\ x' & t'_2 & \\ z' & y' & t'_3 \end{bmatrix} = \begin{bmatrix} t_1t'_1 & & \\ xt'_1 + t_2x' & t_2t'_2 & \\ zt'_1 + \delta yx' + t_3z' & yt'_2 + t_3y' & t_3t'_3 \end{bmatrix},$$

which is almost the usual matrix multiplication except for the twist by δ at the $(3, 1)$ -entry. It will be useful in the sequel to have the following description of B_R^δ modulo its center.

Proposition 2.2. Let R be a commutative ring with identity and $\delta \in R$. Let

$$E_R^\delta = R^2 \rtimes_\delta \begin{bmatrix} R^\times & \\ R & R^\times \end{bmatrix},$$

with the action

$$(2.4) \quad \begin{bmatrix} \alpha & \\ \beta & \gamma \end{bmatrix} \circ_\delta \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} \alpha x \\ \beta x \delta + \gamma z \end{bmatrix}.$$

Then $B_R^\delta / Z_{B_R^\delta} \simeq E_R^\delta$.

Proof. The explicit isomorphism is

$$(2.5) \quad \phi : \begin{bmatrix} t_1 & & \\ x & t_2 & \\ z & y & t_3 \end{bmatrix} \mapsto t_1^{-1} \begin{bmatrix} x \\ z \end{bmatrix} \rtimes t_1^{-1} \begin{bmatrix} t_2 & \\ y & t_3 \end{bmatrix}$$

□

Let $R = \mathcal{O}/\mathcal{P}^m = \mathcal{O}_m$ and in this case denote the groups above H_m^δ , B_m^δ and E^δ . Let T_m stand for the subgroup of diagonal matrices in B_m^δ . The relevance of the groups B_m^δ and E^δ to the permutation representations U_c^m comes from the following.

Theorem 2.3. *For all $m \in \mathbb{N}$ and $c = (c_1, c_2, c_3) \in \mathcal{C}$ the following hold.*

- (a) *The action of P_c on $P_c/P_{c+m\rho}$ factors through B_m^δ with $\delta = \pi^{c_1+c_2-c_3}$.*
- (b) *As B_m^δ -spaces $P_c/P_{c+m\rho} \simeq B_m^\delta/T_m$. In particular, for $c \in C + m\rho$ we have*

$$U_c^m \simeq \text{Ind}_{T_m}^{B_m^\delta}(\mathbf{1}),$$

as B_m^δ -representations.

Before proving Theorem 2.3 we setup notation and make some preparation. Let \mathbf{e}_{ij} denote the matrix whose $(i, j)^{\text{th}}$ entry equal one and the rest of entries are zero. For $i \neq j$ and $x \in R$ (a commutative ring) let

$$(2.6) \quad \begin{aligned} \mathbf{u}_{ij}(x) &= \mathbf{I} + x\mathbf{e}_{ij}, & (\text{elementary unipotent}), \\ \mathbf{s}_i(x) &= \mathbf{I} + (x - 1)\mathbf{e}_{ii}, & (\text{elementary semisimple}). \end{aligned}$$

The following are well known (and easily verified).

$$(2.7) \quad \mathbf{u}_{ij}(x)\mathbf{u}_{kl}(y)\mathbf{u}_{ij}(x)^{-1} = \mathbf{I} + y\mathbf{e}_{kl} + \begin{cases} 0, & \text{if } i \neq l, j \neq k; \\ -xy\mathbf{e}_{kj}, & \text{if } i = l, j \neq k; \\ xy\mathbf{e}_{il}, & \text{if } i \neq l, j = k; \\ xy(\mathbf{e}_{ll} - \mathbf{e}_{kk}) - x^2y\mathbf{e}_{kl}, & \text{if } i = l, j = k. \end{cases}$$

for all $x, y \in R$, $i \neq j$ and $k \neq l$.

$$(2.8) \quad [\mathbf{u}_{ij}(x), \mathbf{s}_k(y)] = \begin{cases} \mathbf{I}, & \text{if } i \neq k, j \neq k; \\ \mathbf{u}_{ij}(x(1 - y)), & \text{if } i = k; \\ \mathbf{u}_{ij}(x(1 - y^{-1})), & \text{if } j = k, \end{cases}$$

for all $x \in R$, $y \in R^\times$. For $c \in \mathcal{C}$ let

$$N_c = \begin{bmatrix} 1 & & \\ \mathcal{P}^{c_1} & 1 & \\ \mathcal{P}^{c_3} & \mathcal{P}^{c_2} & 1 \end{bmatrix}, N^+ = \begin{bmatrix} 1 & \mathcal{O} & \mathcal{O} \\ & 1 & \mathcal{O} \\ & & 1 \end{bmatrix}, T^m = T \cap K^m.$$

Lemma 2.4. *For all $m \in \mathbb{N}$ and $c \in \mathcal{C}$ the following hold.*

- (1) $P_c = N_c T N^+$.
- (2) $N_{c+m\rho} \triangleleft N_c$.
- (3) $[T^m, N_c] \subset N_{c+m\rho}$.

Proof.

- (1) The map $N_c \times T \times N^+ \rightarrow P_c$ given by $(n, t, n^+) \mapsto ntn^+$ is a bijection.

- (2) Since both groups are generated by elementary unipotents (2.6) it is enough to check that $gng^{-1} \in N_{c+m\rho}$ for elementary unipotents $g \in N_c$, $n \in N_{c+m\rho}$. There are only two pairs of such elements which do not commute and for them we verify using (2.7) that

$$\begin{aligned} \mathbf{u}_{21}(\pi^{c_1}x)\mathbf{u}_{32}(\pi^{c_2+m}y)\mathbf{u}_{21}(\pi^{c_1}x)^{-1} &\in \mathbf{I} + \pi^{c_2+m}y\mathbf{e}_{32} - \pi^{c_1+c_2+m}xy\mathbf{e}_{31} \in N_{c+m\rho}, \\ \mathbf{u}_{32}(\pi^{c_2}x)\mathbf{u}_{21}(\pi^{c_1+m}y)\mathbf{u}_{32}(\pi^{c_2}x)^{-1} &= \mathbf{I} + \pi^{c_1+m}y\mathbf{e}_{21} + \pi^{c_1+c_2+m}xy\mathbf{e}_{31} \in N_{c+m\rho}. \end{aligned}$$

- (3) Follows immediately from (2.8). □

Proposition 2.5. *For all $m \in \mathbb{N}$ and $c \in \mathcal{C}^\circ$ the following hold.*

- (1) *As N_c -spaces $P_c/P_{c+m\rho} \simeq N_cT/N_{c+m\rho}T \simeq N_c/N_{c+m\rho}$.*
- (2) *The group $N_{c+m\rho}T^mN^+$ is contained in the kernel of the P_c -action on $P_c/P_{c+m\rho}$.*

Proof. (1) By Lemma 2.4, the group N_c acts transitively on the left cosets of $P_{c+m\rho} = N_{c+m\rho}TN^+$ in $P_c = N_cTN^+$. The stabilizer of $P_{c+m\rho}$ under this action is $N_{c+m\rho}$. It follows that, as N_c -spaces, $P_c/P_{c+m\rho} \simeq N_c/N_{c+m\rho}$. This in particular implies that every orbit in $P_c/P_{c+m\rho}$ can be represented as $nN_{c+m\rho}TN^+$ for some $n \in N_c/N_{c+m\rho}$.

(2) It is clear from Lemma 2.4, that both T^m and $N_{c+m\rho}$ act trivially on $N_c/N_{c+m\rho}$. We show that N^+ acts trivially as well. We need to show that $h(nN_{c+m\rho}TN^+) = nN_{c+m\rho}TN^+$ for $h \in N^+$ and $n \in N_c$, or equivalently that $nhn^{-1} \in N_{c+m\rho}TN^+$. Since both groups N_c and N^+ are generated by unipotents it is enough to check that

$$\mathbf{u}_{ij}(x)\mathbf{u}_{kl}(y)\mathbf{u}_{ij}(x)^{-1} \in N_{c+m\rho}TN^+,$$

for all $\mathbf{u}_{ij}(x) \in N_c$ and $\mathbf{u}_{kl}(y) \in N^+$. The possible indices are $(i, j) \in \{(2, 1), (3, 1), (3, 2)\}$ and $(k, l) \in \{(1, 2), (1, 3), (2, 3)\}$. This is straightforward using (2.7). Thus $N_{c+m\rho}T^mN^+$ is contained in the kernel. □

Proof of Theorem 2.3. Using Lemma 2.4 and Proposition 2.5 we have a commutative diagram

$$\begin{array}{ccccccc} N_{c+m\rho} & \times & T^m & \times & N^+ & \longrightarrow & N_{c+m\rho}T^mN^+ \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ N_c & \times & T & \times & N^+ & \longrightarrow & P_c \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ N_c/N_{c+m\rho} & \times & T_m & \times & \{\mathbf{I}\} & \longrightarrow & P_c/N_{c+m\rho}T^mN^+ \\ \wr & & \parallel & & \parallel & & \wr \\ H_m^\delta & \times & T_m & \times & \{\mathbf{I}\} & \longrightarrow & B_m^\delta, \end{array}$$

with vertical maps being embeddings or canonical epimorphisms and horizontal bijective maps given by multiplication. The identification of the last two rows is induced by the map $\eta : P_c \rightarrow (B_m^\delta, \circ_\delta)$ defined by

$$(2.9) \quad \eta : \begin{bmatrix} t_1 & \star & \star \\ \pi^{c_1}x & t_2 & \star \\ \pi^{c_3}z & \pi^{c_2}y & t_3 \end{bmatrix} \mapsto \begin{bmatrix} t_1 & & \\ x & t_2 & \\ z & y & t_3 \end{bmatrix} \pmod{\mathcal{P}^m},$$

which is an epimorphism with kernel $N_{c+m\rho}T^mN^+$. Assertions (a) and (b) follow. □

For $r = (r_1, r_2, r_3) \in \mathbb{N}_0^3$ with $r_3 \leq r_1 + r_2$ let

$$(2.10) \quad N_r^\delta = \begin{bmatrix} 1 & & \\ \mathcal{P}^{r_1} & 1 & \\ \mathcal{P}^{r_3} & \mathcal{P}^{r_2} & 1 \end{bmatrix} \bmod \mathcal{P}^m, \quad Q_r^\delta = \begin{bmatrix} \mathcal{O}^\times & & \\ \mathcal{P}^{r_1} & \mathcal{O}^\times & \\ \mathcal{P}^{r_3} & \mathcal{P}^{r_2} & \mathcal{O}^\times \end{bmatrix} \bmod \mathcal{P}^m$$

considered as a subgroup of B_m^δ . As N_r^δ does not intersect the center of B_m^δ we also let N_r^δ denote its image in E^δ . Let Θ denote the image of T_m in E^δ .

Corollary 2.6. *For all $m \in \mathbb{N}$, $c \in C^\circ \cap C + m\rho$ and d such that $c - m\rho \preccurlyeq d \preccurlyeq c$ we have*

$$U_d^m \simeq \text{Ind}_{Q_{d-c+m\rho}^\delta}^{B_m^\delta}(\mathbf{1}),$$

as B_m^δ -representations.

$$U_d^m \simeq \text{Ind}_{N_{d-c+m\rho}^\delta}^{E^\delta}(\mathbf{1}),$$

as E^δ -representations. In particular, $U_c^m \simeq \text{Ind}_\Theta^{E^\delta}(\mathbf{1})$.

Proof. The first part follows from the proof of Theorem 2.3 observing that $\eta(P_d) = Q_{d-c+m\rho}^\delta$. The second part follows by observing that the center of B_m^δ acts trivially. \square

3. ON THE PERMUTATION REPRESENTATION $\mathbb{C}[E^\delta/\Theta]$

Let $E^\delta = A \rtimes_\delta \Gamma$, with

$$A = \begin{bmatrix} \mathcal{O}_m \\ \mathcal{O}_m \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \mathcal{O}_m^\times & \\ \mathcal{O}_m & \mathcal{O}_m^\times \end{bmatrix},$$

and the action given by (2.4). Let Θ stand for diagonal matrices in Γ

$$\Theta = \begin{bmatrix} \mathcal{O}_m^\times & \\ & \mathcal{O}_m^\times \end{bmatrix}.$$

The aim of this section is to construct and analyze certain sub-representations of the permutation representation $\text{Ind}_\Theta^{E^\delta}(\mathbf{1}) = \mathbb{C}[E^\delta/\Theta]$. To simplify notation we identify A and Γ with their images in E^δ . We start with the permutation representation of $A\Theta$ on $\mathbb{C}[A\Theta/\Theta]$. The group $A\Theta$ is isomorphic to the semidirect product $(\mathcal{O}_m^2) \rtimes (\mathcal{O}_m^\times)^2$ with the action $a \mapsto \theta a$ for $a \in A, \theta \in \Theta$. Let $\psi : \mathcal{O}_m \rightarrow \mathbb{C}^\times$ be a character which establishes the self duality $\mathcal{O}_m \simeq \widehat{\mathcal{O}_m}$, that is, every character of \mathcal{O}_m is of the form $\psi_\xi : x \mapsto \psi(\xi x)$ for some $\xi \in \mathcal{O}_m$. The characters of A are parameterized by pairs $a = (\xi, \zeta) \in \mathcal{O}_m^2$ which correspond to the character $\varphi_{\xi, \zeta} : (x, z) \mapsto \psi(\xi x + \zeta z)$. Using the induced action of Θ on \widehat{A} we see that the orbits of Θ on \widehat{A} are

$$\Omega_{ij} = \{\varphi_{\xi, \zeta} \mid \text{val}(\xi) = i, \text{val}(\zeta) = j\}, \quad 0 \leq i, j \leq m.$$

Fix $0 \leq i, j \leq m$ and $\varphi \in \Omega_{ij}$. The stabilizer of φ in Θ is

$$\begin{aligned} \Theta_{ij} &= \text{Stab}_\Theta(\varphi) = \{\text{diag}(\theta_1, \theta_2) \mid \theta_1 \equiv_{\pi^{m-i}} 1, \theta_2 \equiv_{\pi^{m-j}} 1\} \\ &\simeq 1 + \pi^{m-i} \mathcal{O}_m \times 1 + \pi^{m-j} \mathcal{O}_m, \end{aligned}$$

and we can extend φ to a character of $A\Theta_{ij}$ using the trivial representation of Θ_{ij} . We denote this extension by φ' . Let

$$(3.1) \quad \mathcal{W}_{ij} = \bigoplus_{\varphi \in \Omega_{ij}} \varphi'.$$

It follows that \mathcal{W}_{ij} is an irreducible representation of $A\Theta$ which is isomorphic to the induced representation $\text{Ind}_{A\Theta_{ij}}^{A\Theta}(\omega')$ for every $\omega \in \Omega_{ij}$. It is the unique irreducible representation of A lying above the orbit Ω_{ij} which has a Θ -fixed vector. Indeed, since $\Theta \backslash A\Theta / A\Theta_{ij}$ is a singleton, the space

$$\text{Hom}_{A\Theta}(\mathcal{W}_{ij}, \text{Ind}_{\Theta}^{A\Theta}(\mathbf{1})) \simeq \{f : A\Theta \rightarrow \mathbb{C} \mid f(\theta gh) = \mathbf{1}(\theta)f(g)\varphi'(h), \forall \theta \in \Theta, \forall g \in A\Theta, \forall h \in A\Theta_{ij}\}$$

is at most one-dimensional. On the other hand a non-trivial intertwining function does exist since $\varphi = \mathbf{1}$ on $A\Theta_{ij} \cap \Theta = \Theta_{ij}$. We record the last discussion in

Lemma 3.1. *There exists decomposition to irreducible $A\Theta$ -representations*

$$\text{Ind}_{\Theta}^{A\Theta}(\mathbf{1}) = \bigoplus_{0 \leq i, j \leq m} \mathcal{W}_{ij},$$

where \mathcal{W}_{ij} is the unique irreducible representation of $A\Theta$ which contains a Θ -fixed vector and whose restriction to A is the sum of the characters in the orbit Ω_{ij} . In particular, $\mathcal{W}_{ij} \simeq \text{Ind}_{A\Theta_{ij}}^{A\Theta}(\varphi')$ for every $\varphi \in \Omega_{ij}$, and

$$\dim \mathcal{W}_{ij} = |\Omega_{ij}| = |\pi^i \mathcal{O}_m^\times| |\pi^j \mathcal{O}_m^\times| = q^{2m-2-i-j}(q-1)^2.$$

We now induce the representations \mathcal{W}_{ij} further to E^δ , and define

$$(3.2) \quad \tilde{\mathcal{W}}_{ij} = \text{Ind}_{A\Theta}^{E^\delta}(\mathcal{W}_{ij}).$$

Our next goal is to find the irreducible components of $\tilde{\mathcal{W}}_{00}$.

Lemma 3.2. *Let $\varphi_{\xi, \zeta} \in \Omega_{00}$ and denote $\epsilon = \delta \xi^{-1} \zeta$. The stabilizer of $\varphi_{\xi, \zeta}$ in Γ is equal to Δ_m^ϵ , where*

(a) *If $\delta \in \mathcal{O}_m^\times$*

$$\Delta_m^\epsilon = \left\{ \begin{bmatrix} \alpha & \\ \beta & 1 \end{bmatrix} \mid \alpha \in \mathcal{O}_m^\times, \beta = \epsilon^{-1}(1 - \alpha) \right\} \simeq \mathcal{O}_m^\times.$$

(b) *If $\delta \in \pi \mathcal{O}_m$*

$$\Delta_m^\epsilon = \left\{ \begin{bmatrix} \alpha & \\ \beta & 1 \end{bmatrix} \mid \beta \in \mathcal{O}_m, \alpha = 1 - \epsilon \beta \right\} \simeq \mathcal{O}_m.$$

Proof. Using the action (2.4) we have

$$\begin{bmatrix} \alpha & \\ \beta & \gamma \end{bmatrix} \varphi_{\xi, \zeta} = \varphi_{\alpha \xi + \delta \beta \zeta, \gamma \zeta},$$

hence the stabilizer consists of elements $\begin{bmatrix} \alpha & \\ \beta & \gamma \end{bmatrix}$ whose entries solve the equations

$$\xi = \alpha \xi + \delta \beta \zeta$$

$$\zeta = \gamma \zeta.$$

The solution depends on δ being a unit or not and is given by cases (a) and (b). □

Theorem 3.3. *The representation $\tilde{\mathcal{W}}_{00}$ of E^δ is multiplicity free with equidimensional irreducible constituents. More precisely, $\tilde{\mathcal{W}}_{00}$ has a decomposition*

$$\tilde{\mathcal{W}}_{00} \simeq \bigoplus_{\sigma \in \Sigma} L_\sigma,$$

where

$$\Sigma = \begin{cases} \widehat{\mathcal{O}_m^\times}, & \text{if } \delta \in \mathcal{O}_m^\times; \\ \widehat{\mathcal{O}_m}, & \text{if } \delta \in \pi\mathcal{O}_m. \end{cases}$$

The representations L_σ are irreducible, non-equivalent and are induced from the one-dimensional extensions of $\varphi = \varphi_{1,1}$ to $\text{Stab}_{E^\delta}(\varphi) = A\Delta_m^\delta$. In particular, their dimension is

$$\dim L_\sigma = [E^\delta : \text{Stab}_{E^\delta}(\varphi)] = \begin{cases} q^{2m-1}(q-1), & \text{if } \delta \in \mathcal{O}_m^\times; \\ q^{2m-2}(q-1)^2, & \text{if } \delta \in \pi\mathcal{O}_m. \end{cases}$$

Proof. Using Lemma 3.1 we have $\tilde{\mathcal{W}}_{00} \simeq \text{Ind}_A^{E^\delta}(\varphi)$ for any $\varphi = \varphi_{\xi,\zeta}$ with $\xi, \zeta \in \mathcal{O}_m^\times$ which we may specify as $\xi = \zeta = 1$. By Lemma 3.2 we have $\text{Stab}_{E^\delta}(\varphi) = A\Delta_m^\delta$. Being a semidirect product, the characters of $A\Delta_m^\delta$ which extend φ are of the form $\varphi\sigma$ where $\sigma \in \Sigma = \widehat{\Delta_m^\delta}$. It follows that

$$(3.3) \quad L_\sigma = \text{Ind}_{A\Delta_m^\delta}^{E^\delta}(\varphi\sigma), \quad (\sigma \in \Sigma),$$

are irreducible and distinct. By Clifford's theorem, the representations L_σ are precisely the irreducible constituents of $\text{Ind}_A^{E^\delta}(\varphi)$, perhaps with multiplicities. However, their direct sum is of dimension $[E^\delta : A]$, therefore, they occur with multiplicity one. \square

4. AN EMBEDDING

Proposition 4.1. *Let $m \in \mathbb{N}$ and $c = (c_1, c_2, c_3) \in C^\circ \cap C + m\rho$. Then, for $0 \leq i, j \leq m$, we have $\tilde{\mathcal{W}}_{ij} \subset U_d^m$ if one (or both) of the following hold.*

- (a) $i > 0$ and $d = (c_1 - 1, c_2, c_3)$.
- (b) $j > 0$ and $d = (c_1, c_2, c_3 - 1)$.

Proof. Let i, j and d satisfy (a) or (b). Note that the assumptions on c and m imply that $d \in C$ and that $c - m\rho \prec d \prec c$. Let $r = d - (c - m\rho)$. By Lemma 3.1 and Corollary 2.6 we have

$$\begin{aligned} \tilde{\mathcal{W}}_{ij} &\simeq \text{Ind}_{A\Theta}^{E^\delta}(\mathcal{W}_{ij}) \simeq \text{Ind}_{A\Theta}^{E^\delta} \left(\text{Ind}_{A\Theta_{ij}}^{A\Theta}(\varphi'_{\xi,\zeta}) \right), \\ U_d^m &\simeq \text{Ind}_{A\Theta}^{E^\delta} \left(\text{Ind}_{N_r^\delta\Theta}^{A\Theta}(\mathbf{1}) \right), \end{aligned}$$

for some $(\xi, \zeta) \in \pi^i\mathcal{O}_m^\times \times \pi^j\mathcal{O}_m^\times$. Therefore, it is enough to show that $\mathcal{W}_{ij} \subset \text{Ind}_{N_r^\delta\Theta}^{A\Theta}(\mathbf{1})$. Since \mathcal{W}_{ij} is irreducible the latter is equivalent to

$$\begin{aligned} 1 &= \dim_{\mathbb{C}} \text{Hom}_A \left(\text{Ind}_{A\Theta_{ij}}^{A\Theta}(\varphi'_{\xi,\zeta}), \text{Ind}_{N_r^\delta\Theta}^{A\Theta}(\mathbf{1}) \right) \\ &= \dim_{\mathbb{C}} \{ f : A\Theta \rightarrow \mathbb{C} \mid f(h_1gh_2) = \mathbf{1}(h_1)f(g)\varphi'_{\xi,\zeta}(h_2), \text{ for all } h_1 \in N_r^\delta\Theta, g \in A\Theta, h_2 \in A\Theta_{ij} \}. \end{aligned}$$

Since $(A\Theta_{ij})(N_r^\delta\Theta) = A\Theta$, the only candidate for the support of a non-zero intertwining function is the identity element. To see that such non-zero function exists we need to check that $\mathbf{1} = \varphi'_{\xi,\zeta}$ on $A\Theta_{ij} \cap N_r^\delta\Theta$. Indeed, for a general element in this intersection we have

$$\varphi_{\xi,\zeta} \left(\begin{bmatrix} x \\ 0 \end{bmatrix} \rtimes \begin{bmatrix} \theta_1 & \\ & \theta_2 \end{bmatrix} \right) = \psi(\xi x) = 1,$$

for case (a) of the proposition, since $x \in \pi^{m-1}\mathcal{O}_m$ and $\xi \in \pi\mathcal{O}_m$. Similarly

$$\varphi_{\xi,\zeta} \left(\begin{bmatrix} 0 \\ z \end{bmatrix} \rtimes \begin{bmatrix} \theta_1 & \\ & \theta_2 \end{bmatrix} \right) = \psi(\zeta z) = 1,$$

for case (b) of the proposition, since $z \in \pi^{m-1}\mathcal{O}_m$ and $\zeta \in \pi\mathcal{O}_m$. \square

Theorem 4.2. *The representation V_c^m is a subrepresentation of $\tilde{\mathcal{W}}_{00}$ for all $m \in \mathbb{N}$ and $c \in C^\circ \cap C + m\rho$. In particular, V_c^m is multiplicity free.*

Proof. By Lemma 3.1 and (3.2) we have

$$U_c^m = \bigoplus_{0 \leq i, j \leq m} \tilde{\mathcal{W}}_{ij},$$

and by Proposition 4.1

$$\sum_{c-m\rho \preceq d \prec c} U_d^m \supset \sum_{(i,j) \neq (0,0)} \tilde{\mathcal{W}}_{i,j}.$$

These together imply that $V_c^m = U_c^m / \sum_{c-m\rho \preceq d \prec c} U_d^m$ embeds in $\tilde{\mathcal{W}}_{0,0}$. \square

5. A DECOMPOSITION

Combining Theorems 3.3 and 4.2 we obtain a list of candidate irreducible subrepresentations of V_c^m ($c \in C^\circ$). In this section we pin down those irreducible representations that indeed occur. For $m \in \mathbb{N}$ let Δ_m^δ be as in Lemma 3.2, namely, isomorphic to \mathcal{O}_m^\times or \mathcal{O}_m depending on whether δ is invertible or not, respectively. To simplify notation we choose such isomorphisms and identify Δ_m^δ with either \mathcal{O}_m^\times or \mathcal{O}_m . Set Δ_0^δ to be the trivial group. Let $\iota : \Delta_m^\delta \rightarrow \Delta_{m-1}^\delta$ denote reduction modulo π^{m-1} . The map ι induces an embedding of characters $\iota^* : \widehat{\Delta}_{m-1}^\delta \hookrightarrow \widehat{\Delta}_m^\delta$. The image consists of characters of Δ_m^δ which factor through Δ_{m-1}^δ . Recall that for $\varphi = \varphi_{1,1}$ we have $\text{Stab}_{E^\delta}(\varphi) \simeq A \rtimes \Delta_m^\delta$, and that for $\sigma \in \widehat{\Delta}_m^\delta$

$$L_\sigma = \text{Ind}_{A\Delta_m^\delta}^{E^\delta}(\varphi\sigma).$$

We also recall that

$$U_d^m \simeq \text{Ind}_{Q_{d-c+m\rho}}^{E^\delta}(\mathbf{1}),$$

and that

$$Q_r^\delta = \begin{bmatrix} \mathcal{P}^{r_1} \\ \mathcal{P}^{r_3} \end{bmatrix} \rtimes \begin{bmatrix} \mathcal{O}_m^\times & \\ \mathcal{P}^{r_2} & \mathcal{O}_m^\times \end{bmatrix}.$$

Theorem 5.1. *For $\sigma \in \widehat{\Delta}_m^\delta$ we have*

$$\text{Hom}_{E^\delta}(L_\sigma, V_c^m) = \begin{cases} 0, & \text{if } \sigma \in \iota^*(\widehat{\Delta}_{m-1}^\delta); \\ 1, & \text{otherwise.} \end{cases}$$

Proof. We show that $L_\sigma \leq \sum_{c-m\rho \preceq d \prec c} U_d^m$ if and only if $\sigma \in \iota^*(\widehat{\Delta}_{m-1}^\delta)$. This is equivalent to the validity of

$$\begin{aligned} J(S): \quad & \text{Hom}_{E^\delta}(L_\sigma, U_d^m) \neq \{0\} \text{ for some } d \in S \text{ if } \sigma \in \iota^*(\widehat{\Delta}_{m-1}^\delta) \\ & \text{Hom}_{E^\delta}(L_\sigma, U_d^m) = \{0\} \text{ for all } d \in S \text{ if } \sigma \notin \iota^*(\widehat{\Delta}_{m-1}^\delta) \end{aligned}$$

for $S = \{d \mid c - m\rho \preceq d \prec c\}$. Since each of the representations U_d^m ($d \in S$) is contained in one of the three maximal representations indexed by $d \in S' = \{c - \mathbf{e}_1, c - \mathbf{e}_2, c - \mathbf{e}_3\}$, with $\{\mathbf{e}_i\}$ denoting the standard basis, it is enough to prove that $J(S')$ holds. Using Corollary 2.6 and the definition of L_σ we have that for $d \in S'$ and $r = d - (c - m\rho)$

$$\begin{aligned} \text{Hom}_{E^\delta}(L_\sigma, U_d^m) &= \text{Hom}_{E^\delta}\left(\text{Ind}_{A\Delta_m^\delta}^{E^\delta}(\varphi\sigma), \text{Ind}_{Q_r^\delta}^{E^\delta}(\mathbf{1})\right) \\ &= \{f : E^\delta \rightarrow \mathbb{C} \mid f(h_1gh_2) = f(g)\varphi\sigma(h_2), \forall h_1 \in Q_r^\delta, \forall g \in E^\delta, \forall h_2 \in A\Delta_m^\delta\}, \end{aligned}$$

and this space is null if and only if $\varphi\sigma \neq {}^g\mathbf{1} = \mathbf{1}$ on $A\Delta_m^\delta \cap {}^gQ_r^\delta$ for all $g \in E^\delta$. We first observe that for $r = m\rho - \mathbf{e}_3 = (m, m, m-1)$ this is indeed the case since the subgroup

$$N_{m\rho - \mathbf{e}_3}^\delta = \begin{bmatrix} 0 \\ \pi^{m-1}\mathcal{O}_m \end{bmatrix} \rtimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \subset A\Delta_m^\delta \cap N_{m\rho - \mathbf{e}_3}^\delta \Theta$$

is a characteristic subgroup in E^δ , and φ is nontrivial on it. Since this is true for any σ the proof is reduced to the validity of $J(S'')$ with $S'' = \{c - \mathbf{e}_1, c - \mathbf{e}_2\}$. A straightforward computation shows that for the corresponding values of r , and regardless of the value of δ , the elements

$$g(y) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rtimes \begin{bmatrix} 1 & \\ y & 1 \end{bmatrix}, y \in \mathcal{O}_m,$$

form an exhaustive set of representatives for the double coset space $Q_r^\delta \backslash E^\delta / A\Delta_m^\delta$. Conjugating Q_r^δ with $g(y)$ gives

$$\begin{aligned} (5.1) \quad & \left\{ \begin{bmatrix} x \\ \delta xy \end{bmatrix} \rtimes \begin{bmatrix} \theta_1 & \\ (\theta_1 - \theta_2)y & \theta_2 \end{bmatrix} \mid \theta_1, \theta_2 \in \mathcal{O}_m^\times, x \in \mathcal{P}^{m-1} \right\}, \quad \text{if } r = m\rho - \mathbf{e}_1; \\ & \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rtimes \begin{bmatrix} \theta_1 & \\ u + (\theta_1 - \theta_2)y & \theta_2 \end{bmatrix} \mid \theta_1, \theta_2 \in \mathcal{O}_m^\times, u \in \mathcal{P}^{m-1} \right\}, \quad \text{if } r = m\rho - \mathbf{e}_2. \end{aligned}$$

We now proceed according to the value of δ .

(a) $\delta \in \mathcal{O}_m^\times$.

The intersection ${}^{g(y)}Q_r^\delta \cap A\Delta_m^\delta$ is given by

$$\begin{aligned} & \left\{ \begin{bmatrix} x \\ \delta xy \end{bmatrix} \rtimes \begin{bmatrix} \theta & \\ (\theta - 1)y & 1 \end{bmatrix} \mid \theta \in 1 + \mathcal{P}^{m-\text{val}(y\delta+1)}, x \in \mathcal{P}^{m-1} \right\}, \quad \text{if } r = m\rho - \mathbf{e}_1; \\ & \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rtimes \begin{bmatrix} \theta & \\ u + (\theta - 1)y & 1 \end{bmatrix} \mid \theta \in 1 + \mathcal{P}^{m-\text{val}(y\delta+1)-1}, u = (1 - \theta)(1 + \delta y)\delta^{-1} \right\}, \quad \text{if } r = m\rho - \mathbf{e}_2. \end{aligned}$$

If $r = m\rho - \mathbf{e}_1$, then for a given σ the element $g(y)$ supports a nonzero intertwiner if and only if

$$\varphi \begin{bmatrix} x \\ \delta xy \end{bmatrix} \sigma(\theta) = \psi(x(1+y\delta)) \sigma(\theta) = 1, \quad \forall x \in \mathcal{P}^{m-1}, \forall \theta \in 1 + \mathcal{P}^{m-\text{val}(y\delta+1)},$$

that is, $\text{val}(1+y\delta) > 0$ and $\sigma|_{1+\mathcal{P}^{m-\text{val}(y\delta+1)}} = \mathbf{1}$. Therefore, $L_\sigma \leq U_{c-\mathbf{e}_1}^m$ if and only if $\sigma|_{1+\mathcal{P}^{m-1}} = \mathbf{1}$.

If $r = m\rho - \mathbf{e}_2$, then for a given σ the element $g(y)$ supports a nonzero intertwiner if and only if

$$\varphi \begin{bmatrix} 0 \\ 0 \end{bmatrix} \sigma(\theta) = 1, \quad \forall \theta \in 1 + \mathcal{P}^{m-\text{val}(y\delta+1)-1},$$

that is, $\sigma|_{1+\mathcal{P}^{m-\text{val}(y\delta+1)-1}} = \mathbf{1}$. Therefore, $L_\sigma \leq U_{c-\mathbf{e}_1}^m$ if and only if $\sigma|_{1+\mathcal{P}^{m-1}} = \mathbf{1}$.

(b) $\delta \in \pi\mathcal{O}_m$.

The intersection ${}^{g(y)}Q_{m\rho-\mathbf{e}_1}^\delta \cap A\Delta_m^\delta$ is given by

$$\left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \rtimes \begin{bmatrix} \theta & \\ (\theta-1)y & 1 \end{bmatrix} \mid \theta \in 1 + \mathcal{P}^{m-\text{val}(y\delta+1)}, x \in \mathcal{P}^{m-1} \right\}, \quad \text{if } r = m\rho - \mathbf{e}_1;$$

$$\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rtimes \begin{bmatrix} 1 & \\ u & 1 \end{bmatrix} \mid u \in \mathcal{P}^{m-1} \right\}, \quad \text{if } r = m\rho - \mathbf{e}_2.$$

If $r = m\rho - \mathbf{e}_1$, then

$$\varphi \begin{bmatrix} x \\ 0 \end{bmatrix} \neq \mathbf{1},$$

and we conclude that $L_\sigma \not\leq U_{c-\mathbf{e}_1}^m$ for all σ 's.

If $r = m\rho - \mathbf{e}_2$, then if $\sigma|_{\mathcal{P}^{m-1}} = \mathbf{1}$ in fact all the elements $g(y)$ support a non-zero intertwining function and none of them otherwise. Therefore, $L_\sigma \leq U_{c-\mathbf{e}_1}^m$ if and only if $\sigma|_{\mathcal{P}^{m-1}} = \mathbf{1}$.

The theorem is proved. □

Definition 5.2. Let $c \in C^\circ$ and let $\delta = \pi^{\kappa(c)-\mu(c)}$. For $\sigma \in \widehat{\Delta}_{\mu(c)}^\delta \setminus \iota^* \left(\widehat{\Delta}_{\mu(c)-1}^\delta \right)$ let

$$\widetilde{L}_{c,\sigma} = \text{Ind}_{P_{c-\mu(c)\rho}}^G \left(\text{Inf}_{E^\delta}^{P_{c-\mu(c)\rho}} (L_\sigma) \right),$$

where $\text{Inf}_{E^\delta}^{P_{c-\mu(c)\rho}} (L_\sigma)$ stands for the pullback of L_σ along the composition of the quotient maps (2.5) and (2.9).

Corollary 5.3. For $c \in C^\circ$ the decomposition of V_c to irreducible representations is multiplicity free and given by

$$V_c = \bigoplus_{\sigma \in \widehat{\Delta}_{\mu(c)}^\delta \setminus \iota^* \left(\widehat{\Delta}_{\mu(c)-1}^\delta \right)} \widetilde{L}_{c,\sigma}.$$

Proof. Follows from Theorem 5.1 and statement (CN3). □

6. MULTIPLICITIES AND DEGREES

In this section we compute explicitly the dimensions and multiplicities of the irreducible constituents in $V = \text{Ind}_B^G(\mathbf{1})$.

6.1. Multiplicities. We first settle a small debt from subsection 1.2. we shall need the following lemma.

Lemma 6.1. *Let $x, y, z \in \mathbb{N}$ be such that $0 < x + y - z < \min\{x, y\}$. Then the following are equivalent*

- (1) $x + y - z \leq \lfloor \min\{x, y\}/2 \rfloor$.
- (2) $\mu(x, y, z) = \min\{x + y - z, z - x, z - y\} = x + y - z = \kappa(x, y, z)$.

Proof. Without loss of generality assume that $x \leq y$. Then (1) is equivalent to $2(x + y - z) \leq x \leq y$, which in turn is equivalent to $x + y - z \leq z - y \leq z - x$, and the latter is equivalent to (2). \square

Proposition 6.2. *Let $c, d \in C$. Then $V_c \simeq V_d$ if and only if $\lambda(c) = \lambda(d)$, $\kappa(c) = \kappa(d)$, $\mu(c) = \mu(d)$ and $c = d$ if $\kappa(c) > \mu(c)$.*

Proof. We go over the various possibilities for c .

- (1) If $c \in C$ satisfies $c_3 = c_1 + c_2$, then by Theorem 6.1 and Proposition 6.2 in [1], for any $d \in C$ we have $V_c \simeq V_d$ if and only if $c_3 = d_3$ and $c_1 + c_2 = d_1 + d_2$. These two equations are equivalent to $\lambda(c) = \lambda(d)$ and $\kappa(c) = \kappa(d)$, and in such case $0 \leq \mu(c) \leq \kappa(c) = 0$.
- (2) If $c \in C$ satisfies $c_3 = \max\{c_1, c_2\} \geq 1$, then by [1, Theorem 7.1] and the discussion preceding it, for any $d \in C$ we have $V_c \simeq V_d$ if and only if $c = d$. The conditions on c imply that $\kappa(c) > \mu(c) = 0$ and the assertion follow.
- (3) It remains to treat $c, d \in C^\circ$. In particular this means that $\kappa(c), \kappa(d) > 0$. By [1, Theorem 8.1], we have that $V_c \simeq V_d$ if and only if
 - (i) $\lambda(c) = c_3 = d_3 = \lambda(d)$.
 - (ii) $\kappa(c) = c_1 + c_2 - c_3 = d_1 + d_2 - d_3 = \kappa(d)$.
 - (iii) $c_1 + c_2 - c_3 \leq \lfloor \min\{c_1, c_2, d_1, d_2\}/2 \rfloor$.

In the presence of (i) and (ii), condition (iii) is equivalent to $\mu(c) = \kappa(c) = \kappa(d) = \mu(d)$, by applying Lemma 6.1 to c and to d .

\square

Define an equivalence relation on C by setting $c \sim d$ if and only if $V_c \simeq V_d$. Let $a : \mathbb{N}_0^3 \rightarrow \mathbb{N}_0$ be the function defined by

$$(6.1) \quad a(m, k, \ell) = \begin{cases} \ell - 3k + 1, & \text{if } \ell \geq 3k = 3m \geq 0 ; \\ 1, & \text{if } \ell \geq 2m + k > 3m \geq 0, \end{cases}$$

and zero otherwise.

Proposition 6.3. $|[c]| = a(\mu(c), \kappa(c), \lambda(c))$ for every $c \in C$.

Proof. If $c \in C$ satisfies $\kappa(c) = k = m = \mu(c)$ and $\lambda(c) = c_3 = \ell$ then $k = c_1 + c_2 - \ell \leq \ell - c_1, \ell - c_2$. It follows that $2k \leq c_1, c_2 \leq \ell - k$, in particular $\ell \geq 3k$, and that the c 's satisfying these conditions are precisely of the form

$$\{(2k + i, \ell - k - i, \ell) \mid i = 0, \dots, \ell - 3k\},$$

and their number is $\ell - 3k + 1$.

If $c \in C$ satisfies $\kappa(c) = k > m = \mu(c)$ and $\lambda(c) = c_3 = \ell$, then Proposition 6.2 implies that $V_c \simeq V_d$ if and only if $c = d$ hence $[c]$ is a singleton. We also have that

$$\ell = (c_1 + c_2 - c_3) + (c_3 - c_1) + (c_3 - c_2) \geq k + 2m.$$

□

Corollary 6.4. *A complete decomposition of $V = \text{Ind}_B^G(\mathbf{1})$ to irreducible representations is given by*

$$V \simeq \left(\bigoplus_{c \in \partial C} V_c \right) \oplus \left(\bigoplus_{[c] \in C^\circ / \sim} \bigoplus_{\sigma \in \hat{\Delta}_m^\delta \setminus \iota^* \hat{\Delta}_{m-1}^\delta} \tilde{L}_{c,\sigma}^{\oplus a(m,k,\ell)} \right).$$

Proof. Follows from (1.1), (CN2), Corollary 5.3 and Proposition 6.3.

□

6.2. Dimensions. The equidimensionality of the irreducible constituents of V_c implies that the dimension of an irreducible subrepresentation W of V is uniquely determined by the elements $c \in C$ with $\text{Hom}_G(W, V_c) \neq (0)$. The dimension of an irreducible representation occurring in V_c ($c \in C$) is

$$(6.2) \quad \begin{cases} 1, & \text{if } c = (0, 0, 0); \\ q^2 + q, & \text{if } c = (1, 0, 1), (0, 1, 1); \\ q^3, & \text{if } c = (1, 1, 1); \\ (1 + q^{-1})(1 - q^{-3})q^{2\lambda(c)}, & \text{if } \kappa(c) = \mu(c), \lambda(c) \geq 2; \\ (1 - q^{-2})(1 - q^{-3})q^{2\lambda(c) + \kappa(c) - \mu(c)}, & \text{if } \kappa(c) > \mu(c), \lambda(c) \geq 2. \end{cases}$$

Indeed, the formulae for $\lambda(c) \leq 1$ are well known, see e.g. [1]. For $c \in C^\circ$ with $\lambda(c) \geq 2$ they follow from the construction of $\tilde{L}_{c,\sigma}$:

$$\begin{aligned} \dim \tilde{L}_{c,\sigma} &= [\text{GL}_3(\mathcal{O}) : P_{c-\mu(c)\rho}] \cdot \dim L_\sigma \\ &= (1 + q^{-1})(1 + q^{-1} + q^{-2}) \cdot q^{c_1 + c_2 + c_3 - 3\mu(c)} \cdot \begin{cases} q^{2\mu(c)}(1 - q^{-1}), & \text{if } \kappa(c) = \mu(c); \\ q^{2\mu(c)}(1 - q^{-1})^2, & \text{if } \kappa(c) > \mu(c); \end{cases} \\ &= \begin{cases} (1 + q^{-1})(1 - q^{-3})q^{2\lambda(c)}, & \text{if } \kappa(c) = \mu(c); \\ (1 - q^{-2})(1 - q^{-3})q^{2\lambda(c) + \kappa(c) - \mu(c)}, & \text{if } \kappa(c) > \mu(c). \end{cases} \end{aligned}$$

For $c \in \partial C$ with $\lambda(c) \geq 2$ the formulae are given by Theorems 6.1 and 7.1 in [1].

6.3. Uniform representation growth. Let $\zeta_{G,V}(s) = \sum_{n=1}^\infty r_n(G, V)n^{-s}$ with $r_n(G, V)$ the number of irreducible n -dimensional subrepresentations of V of dimension n . The generating function $\zeta_{G,V}(s)$ enumerates irreducible representations in V according to their dimensions and regardless of their isomorphism type. For every $n \in \mathbb{N}$, let $p_n \in \{0, 1, 2\}$ be the residue of n modulo 3 and let $f_n(x), g_n(x) \in \mathbb{Z}[x]$ be the polynomials

$$\begin{aligned} f_n(x) &= x^{\lfloor n/6 \rfloor - 1}((p_{\frac{n}{2}} + 1)x + (2 - p_{\frac{n}{2}})), \quad (n \geq 4), \\ g_n(x) &= \frac{x^{\lfloor n/2 \rfloor} - 1}{x - 1} + \frac{x^{\lfloor n/2 \rfloor - 1} - 1}{x - 1} + \min\{p_n, 1\}x^{\lfloor n/2 \rfloor - 1}, \quad (n \geq 5). \end{aligned}$$

Denote $\eta_1(q) = (1 + q^{-1})(1 - q^{-3})$ and $\eta_2(q) = (1 - q^{-2})(1 - q^{-3})$.

Theorem 6.5. *For every non-archimedean local field F with ring of integers \mathcal{O} and residue field of cardinality q*

$$\zeta_{G,V}(s) = 1 + 2(q + q^2)^{-s} + q^{-3s} + \sum_{n=4}^{\infty} f_n(q) (\eta_1(q)q^n)^{-s} + \sum_{n=5}^{\infty} g_n(q) (\eta_2(q)q^n)^{-s}.$$

Proof. The first three terms in $\zeta_{G,V}(s)$ correspond to the subrepresentations of V_c with $\lambda(c) \in \{0, 1\}$. For the remaining, we observe that the dimension of an irreducible representation in V with $\lambda(c) \geq 2$, as listed in (6.2), determines whether the representation V_c which contains it has $\kappa(c) = \mu(c)$ or $\kappa(c) > \mu(c)$, hence determines its construction. We formally set $\mathcal{O}_0^\times = \{1\}$ and $\mathcal{O}_0 = \{0\}$ to be the trivial groups and $\mathcal{O}_{-1}^\times = \mathcal{O}_{-1}$ to be the empty set. For the irreducible representations of dimension $\eta_1(q)q^n$ we have

$$\begin{aligned} r_{\eta_1(q)q^{-n}}(G, V) &= \sum_{m: 0 \leq 6m \leq 2\ell = n} a(m, m, \ell) |\mathcal{O}_m^\times \setminus \mathcal{O}_{m-1}^\times| \\ &= \sum_{0 \leq m \leq \lfloor \ell/3 \rfloor} (\ell - 3m + 1) |\mathcal{O}_m^\times| - \sum_{-1 \leq m \leq \lfloor \ell/3 \rfloor - 1} (\ell - 3m - 4) |\mathcal{O}_m^\times| \\ &= (\ell - 3\lfloor \ell/3 \rfloor + 1) |\mathcal{O}_{\lfloor \ell/3 \rfloor}^\times| + 3 \sum_{0 \leq m \leq \lfloor \ell/3 \rfloor - 1} |\mathcal{O}_m^\times| \\ &= q^{\lfloor n/6 \rfloor - 1} \left((p_{\frac{n}{2}} + 1)q + (2 - p_{\frac{n}{2}}) \right), \end{aligned}$$

for $n \in 2\mathbb{N}_0 + 4$ and zero otherwise.

It remains to consider the irreducible representations of dimension $\eta_2(q)q^n$. By (6.2), an the irreducible subrepresentation of V of dimension $\eta_2(q)q^n$ is contained in some V_c with $\kappa(c) > \mu(c)$. For $m, n \in \mathbb{N}_0$ let $S(m, n)$ be the (possibly empty) subset of C

$$S(m, n) = \{c \in C \mid 2\lambda(c) + \kappa(c) - \mu(c) = n, \kappa(c) > \mu(c) = m\}.$$

If $S(m, n)$ is non-empty and $c \in S(m, n)$ we have $a(\mu(c), \kappa(c), \lambda(c)) = 1$, and by Corollary 5.3 the number of irreducible constituents of V_c is equal to $|\mathcal{O}_m \setminus \pi\mathcal{O}_m|$. It follows that the number of irreducible constituents of V of dimension $\eta_2(q)q^n$ is equal to

$$(6.3) \quad \sum_{m, k: n=2\ell+k-m, k>m} a(m, k, \ell) |\mathcal{O}_m \setminus \pi\mathcal{O}_m| = \sum_{m \in \mathbb{N}_0} |S(m, n)| |\mathcal{O}_m \setminus \pi\mathcal{O}_m|.$$

Note that by the definition of $S(m, n)$ this is a finite sum for every $n \in \mathbb{N}$. Unraveling definitions and carrying the necessary book keeping yields, for $n \geq 5$,

$$|S(m, n)| = \begin{cases} 2\lfloor n/2 \rfloor - 2m, & \text{if } 0 \leq m \leq \lfloor n/2 \rfloor - 1, n \not\equiv 0 \pmod{3}; \\ 2\lfloor n/2 \rfloor - 2m + 1, & \text{if } 0 \leq m \leq \lfloor n/2 \rfloor, n \equiv 0 \pmod{3}, \end{cases}$$

and zero otherwise. By substituting these values in (6.3), we see that for $n \equiv 1, 2 \pmod{3}$,

$$(6.4) \quad \sum_{m=0}^{\lfloor n/2 \rfloor - 1} 2(\lfloor n/2 \rfloor - m)(q-1)q^{m-2} = 2 \frac{q^{\lfloor n/2 \rfloor} - 1}{q-1},$$

and for $n \equiv 0 \pmod{3}$,

$$(6.5) \quad \sum_{m=0}^{\lfloor n/2 \rfloor} (2\lfloor n/2 \rfloor - 2m + 1) (q - 1) q^{m-2} = \frac{q^{\lfloor n/2 \rfloor + 1} - 1}{q - 1} + \frac{q^{\lfloor n/2 \rfloor} - 1}{q - 1}.$$

Combining these expressions, we obtain the result. □

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